Banach Contraction Principle

And Bessaga's Converse to BCP

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Abstract

"Although Banach's theorem is quite easy to prove, but a longer proof cannot yield stronger results."
Eric Schechter

In the year 1922 in his PhD thesis, Stefan Banach proved the Banach Contraction Principle (BCP), also regarded as Banach's Fixed Point Theorem, which states that :

> *Let $X \neq \emptyset$, and let (X, d) be a complete metric space. Suppose $f : X \to X$ be a contraction map, i.e., $\exists\, \alpha \in [0, 1)$ such that*
>
> $$d(f(x), f(y)) \leq \alpha d(x, y)\ \forall\, x, y \in X$$
>
> *then f must have an unique fixed point $\xi \in X$. Moreover the function $f^n : X \to X$ defined by $f^n := \underbrace{f \circ \cdots \circ f}_{n \text{ times}}\ \forall\, n \in \mathbb{N}$, also has an unique fixed point at ξ.*

Since then it is widely used in various fields of mathematics and is also regarded as the most important result in Metric Fixed Point Theory. There have been various extensions and converses to the BCP. In this lecture we will see that even though all the extensions and generalizations of BCP are quite useful in some sense, but they do not really improve Banach's theorem! We give a motivation for converse to Banach's theorem. Then we will give an intuitive proof of Bessaga's Converse to BCP using a simple idea from Graph Theory and use it to prove that Banach's theorem cannot be improved further.

Keywords : *Fixed Point, Contraction Map, Lipschitz map, Metric Space, Complete Metric Space, Proper Metric Space, Banach Contraction Principle (BCP), Bessaga Brunner Metric, Bessaga's Converse to BCP, Meyer's Converse to BCP, Daskalakis Tzamos Zampetakis Converse of BCP.*

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1 Notations

Definition 1.0.1 (Fixed Point). *Let $f : X \to X$ then $x_0 \in X$ is said to be a fixed point of f if $f(x_0) = x_0$.*

Definition 1.0.2 (Metric Space). *Let X be a non-empty set and let $d : X^2 \to [0, \infty)$ be such that*

1. $d(x, y) \geq 0, \forall\, x, y \in X$ *and* $d(x, y) = 0$ *iff* $x = y$.
2. $d(x, y) = d(y, x)\ \forall\, x, y \in X$
3. $d(x, z) + d(z, y) \geq d(x, y)\ \forall\, x, y, z \in X$

Then we say X is a metric space with respect to the metric d, which is abbreviated as (X, d).

Definition 1.0.3 (Cauchy Sequence). *Let (X, d) be a metric space, then a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said to be Cauchy sequence with respect to the metric d, if $\forall \varepsilon > 0,\ \exists\, N_\varepsilon \in \mathbb{N}$ such that*

$$d(x_m, x_n) < \varepsilon\ \forall\, m, n \geq N_\varepsilon$$

Definition 1.0.4 (Complete Metric Space). *A metric space (X, d) is said to be complete if every Cauchy sequence in (X, d) converges in X.*

Definition 1.0.5 (Proper Metric Space). *A metric space (X, d) is said to be proper if every closed and bounded subsets in X are compact.*

Definition 1.0.6 (Contraction Map). *Let (X, d) be a metric space and suppose $f : X \to X$ is said to be a contraction map if $\exists\, \alpha \in [0, 1)$ such that*

$$d(f(x), f(y)) \leq \alpha d(x, y)\ \forall\, x, y \in X$$

It should be noted that contraction map are a particular case of Lipschitz functions, whose Lipschitz constant is strictly less than 1. Also note that

Contraction Map $\Rightarrow$ Lipschitz Function $\Rightarrow$ Continuity

$f_{<lips>} = \inf\{\alpha \in [0,1) \ : \ d(f(x), f(y)) \leq \alpha d(x,y) \ \forall \ x, y \in X\}$, is called the *contraction constant* of f, we can define $f_{<lips>}$ also by

$$f_{<lips>} = \sup\left\{\frac{d(f(x), f(y))}{d(x,y)} \ : \ x, y \in X, \ x \neq y\right\}$$

Definition 1.0.7. *Let $f : X \to X$ then n^{th} iterate of f is the function $f^n : X \to X$ defined by*

$$f^n := \underbrace{f \circ \cdots \circ f}_{n \text{ times}} \ \forall \ n \in \mathbb{N}$$

and we let $f^0(x) = x, \ \forall \ x \in X$.

Definition 1.0.8. *Let $f : X \to X$, we will say f is a stable function (or say f satisfies BCP) if every iterate of f has an unique fixed point, i.e.*

$$f^n(x) = x$$

has an unique solution $\xi \in X$ for each positive integer n.

2 Banach Contraction Principle

Theorem 2.0.1. *Let $X \neq \emptyset$ be a complete metric space, with metric d and suppose $f : X \to X$ be a contraction map, then f must satisfy BCP, i.e f is stable.*

Proof *(of Theorem 2.0.1).*

If f is a constant map, then the result is obvious. So we assume that f is not a constant map, thus the contraction constant of f is strictly positive. Let $\alpha \in (0,1)$ be the contraction constant of f then

$$d(f(x), f(y)) \leq \alpha d(x,y) \ \forall \ x, y \in X$$

As $X \neq \emptyset$ we can choose a point $x_0 \in X$. Now define a sequence $\{x_n\}_{n=1}^{\infty}$ recursively by $x_n = f(x_{n-1}) \ \forall \ n \in \mathbb{N}$. Now observe that

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq \alpha d(x_n, x_{n-1}) \leq \alpha^2 d(x_{n-1}, x_{n-2}) \leq \cdots \leq \alpha^n d(x_1, x_0) \quad (1)$$

Now let $\frac{d(x_1,x_0)}{1-\alpha} = C > 0$ and $m > n$ then using (1) we get

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \alpha^{m-1} d(x_1, x_0) + \alpha^{m-2} d(x_1, x_0) + \ldots \alpha^n d(x_1, x_0) \\ &= \alpha^n d(x_1, x_0)\left(\alpha^{m-n-1} + \alpha^{m-n-2} + \cdots + 1\right) \\ &\leq \alpha^n \left(\frac{d(x_1, x_0)}{1-\alpha}\right) = C\alpha^n \end{aligned}$$

Now let $\varepsilon > 0$ then as $\alpha \in (0,1) \Rightarrow \alpha^n \to 0$ as $n \to \infty$ thus $\exists\, N_\varepsilon$ such that $\alpha^n < \frac{\varepsilon}{C}\ \forall\, n \geq N_\varepsilon$ thus $d(x_m, x_n) < \varepsilon\ \forall\, m, n \geq N_\varepsilon$. Thus $\{x_n\}$ is a Cauchy sequence and as (X, d) is a complete metric space we get x_n converges in X. Let $\lim_{n\to\infty} x_n = \xi$ then $f(x_n) = x_{n+1} \Rightarrow \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = \xi$. Now as f is continuous function we get

$$f(\xi) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \xi$$

Also if $\zeta \neq \xi$ is any other fixed point of f then we get that

$$\alpha d(\zeta, \xi) \geq d(f(\zeta), f(\xi)) = d(\zeta, \xi) \Rightarrow \alpha \geq 1 \text{ (Contradiction!)}$$

Thus ξ is the unique fixed point of f. Now to show that ξ is the unique fixed point of every iterate of f, note that

$$d(f^n(x), f^n(y)) \leq \alpha^n d(x, y)\ \forall\, x, y \in X,\ n \in \mathbb{N}$$

thus f^n is a contraction map on X and hence has an unique fixed point. But ξ is already a fixed point of f^n for any positive integer n, thus ξ is the unique fixed point of any iterate of f and hence f satisfies BCP, i.e, f is stable. □

The best way to grasp a theorem is to look into some examples.

Example 2.0.2. *Let $X = (0,1) \subset \mathbb{R}$ and consider the standard Euclidean metric. Let $f : X \to X$ given by*

$$f(x) = \frac{x}{2}\ \forall\, x \in X$$

Then clearly f is a contraction map with respect to standard Euclidean metric, but note that f does not have any unique fixed point in the set X, So f cannot be a stable function.

The observation we can make from this example is that if the metric space is not complete, then the contraction map may not satisfy BCP. But then the question is - Is it completeness always necessary ? Our next example shows that completeness is actually not a necessary condition for BCP to hold.

Example 2.0.3. *Consider the metric space $X = \left(-\frac{1}{3}, \frac{1}{3}\right)$ with respect to standard Euclidean metric and let $f : X \to X$ is given by*

$$f(x) = x^2\ \forall\, x \in X$$

Then observe that f is a contraction map on X with respect to standard Euclidean metric and is also stable as $x = 0$ is the unique fixed point of any iterate of f in the domain X. But clearly X is not a complete metric space !

Remarks

Contraction map on a complete metric space always satisfies Banach Contraction Principle (BCP), but a contraction map on a metric space (X, d) that satisfies BCP doesnot guarantee that the metric space (X, d) is complete, in fact there are various examples of an incomplete metric space in which every contraction map satisfies BCP (see [5]). Thus Banach's theorem cannot characterize metric completeness. But there is an intriguing result due to Dr. PV Subrahmanyam, which shows that Kannan type contraction map characterizes metric completeness (see [9]).

So, now we know that a contraction map on a complete metric space is always stable. Now suppose that we are given a stable function f on a set X, then what conditions on the function f guarantees that there exists a complete metric space (X,d) such that f is a contraction map on the metric space (X,d) ? Basically we are trying to look for a converse statement to Banach Contraction Principle. Let us look at another example that will help us to understand what we really want to convey.

Example 2.0.4. *Consider* $X=(-1,1)$ *and* $f:X\to X$ *given by*

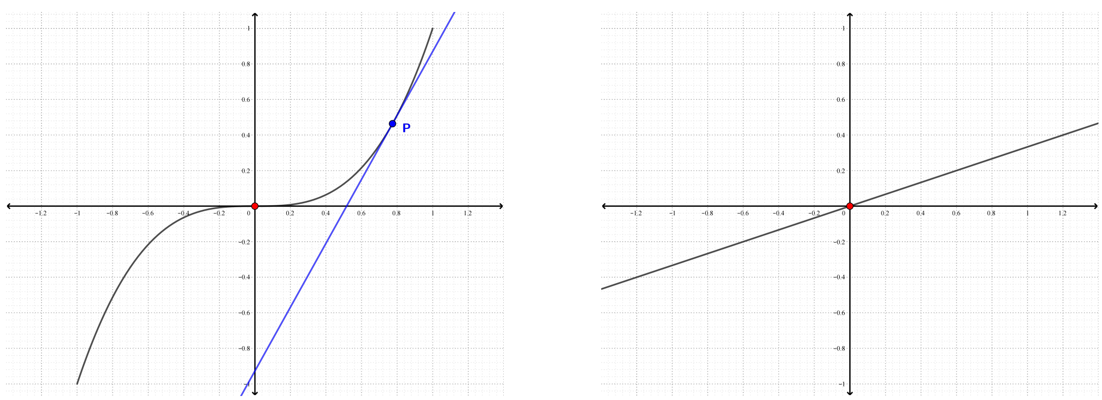
$$f(x)=x^3\ \forall\, x\in X$$

Obviously this function is stable, as $x=0$ *is the unique fixed point of any iterate of* f *in the domain* X*. But this is not a contraction map with respect to standard Euclidean metric, because*

$$f'(x)>1,\ \forall\, x\in\left(-1,-\frac{1}{\sqrt{3}}\right)\cup\left(\frac{1}{\sqrt{3}},1\right)$$

Simply consider the point $\frac{\sqrt{3}}{2}$*, then we have* $f'\left(\frac{\sqrt{3}}{2}\right)=\frac{9}{4}>1$*. So we can find a neighborhood* $\mathcal{U}\subset X$ *of* $\frac{\sqrt{3}}{2}$ *such that*

$$\left|\frac{f(x)-f(y)}{x-y}\right|>1\ \forall\, x,y\in\mathcal{U}$$



(a) *Graph of* f *wrt Euclidean metric* (b) *Graph of* f *wrt metric* d

Figure 1

But if we consider the metric to be

$$d(x,y)=\left|\frac{sgn(x)}{\ln|x|}-\frac{sgn(y)}{\ln|y|}\right|\ \forall\, x,y\in X$$

then we get f *is a contraction map on* (X,d)*, to view this let* $x,y\in X$ *then*

$$\begin{aligned}
d(f(x),f(y)) &= d(x^3,y^3)\\
&= \left|\frac{sgn(x^3)}{\ln|x^3|}-\frac{sgn(y^3)}{\ln|y^3|}\right|\\
&= \frac{1}{3}\left|\frac{sgn(x)}{\ln|x|}-\frac{sgn(y)}{\ln|y|}\right|\\
&= \frac{1}{3}d(x,y)
\end{aligned}$$

Actually if we scale the points on the set X with respect to the metric d, the equation

$$d(f(x), f(y)) = \frac{1}{3} d(x, y) \; \forall \, x, y \in X$$

tells us that the graph of f with respect to the metric d is simply a straight line passing through the origin with slope $\frac{1}{3}$. Interestingly enough it also turns out that the metric space (X, d) is a complete metric space!

Our recent example gives us some hope that for a stable function f defined on a set X we can find a metric d such that f is a contraction map on (X, d) and in fact the metric space (X, d) is complete. This is were Bessaga's theorem comes to light! Bessaga showed that indeed if we are given a stable function, we can always find a complete metric space on which it is a contraction map. Now we are dwelling into the topic of converse of Banach Contraction Principle, and there are in fact various converse to BCP such as Bessaga's converse, Meyer's converse (see [7]) and Daskalakis Tzamos Zampetakis converse (see [4]).

Theorem 2.0.5 (Bessaga's Converse to BCP)**.** *Let X be a non-empty set and $f : X \to X$ is such that $\xi \in X$ is the unique fixed point of f^n for all positive integer n, i.e, f is a stable function. Then for any $c \in (0, 1)$ there exists a metric d_c such that (X, d_c) is a complete metric space and f is a contraction map with contraction constant c with respect to the metric d_c.*

Theorem 2.0.6 (Meyer's Converse to BCP)**.** *Suppose (X, ρ) is a complete metric space where X is compact, $f : X \to X$ is continuous with respect to ρ and the following hold:*

1. *f has a unique fixed point ξ.*
2. *$\forall \, x \in X$ we have $f^n(x) \to \xi$ as $n \to \infty$*
3. *$\exists$ an open neighborhood $\mathcal{U}$ of ξ such that $f^n(\mathcal{U}) \to \{\xi\}$*

Then for any $c \in (0, 1)$ there exists a metric d_c which is topologically equivalent to ρ, such that (X, d_c) is a complete metric space and f is a contraction map with contraction constant c with respect to the metric d_c.

Theorem 2.0.7 (Daskalakis Tzamos Zampetakis)**.** *Suppose (X, ρ) is a complete, proper metric space, $f : X \to X$ is continuous with respect to ρ and the following hold:*

1. *f has a unique fixed point ξ.*
2. *$\forall \, x \in X$ we have $f^n(x) \to \xi$ as $n \to \infty$*
3. *$\exists$ an open neighborhood $\mathcal{U}$ of ξ such that $f^n(\mathcal{U}) \to \{\xi\}$*

Then for every $c \in (0, 1)$ and $\varepsilon > 0$, there exists a metric $d_{c,\varepsilon}$ that is topologically equivalent to ρ and is such that $(X, d_{c,\varepsilon})$ is a complete metric space and

$$\forall \, x, y \in X : d_{c,\varepsilon}(f(x), f(y)) \leq c d_{c,\varepsilon}(x, y)$$
$$\forall \, x, y \in X : d_{c,\varepsilon}(x, y) \leq \varepsilon \Rightarrow \min\{d(\xi, x), d(\xi, y), d(x, y)\} \leq 2\varepsilon$$

3 Bessaga's Converse to BCP

The original proof of Bessaga's theorem uses a special kind of result from *Axiom of Choice* (see [3]). There are various other proofs of Bessaga's converse (see [6]). Over here we present a very elegant proof which is given in [11].

Proof *(of Theorem 2.0.5)*

Let $\Lambda : X \to (0, \infty)$ be a function. Now we define an equivalence relation $\sim$ on the set X, let $x \sim y$ if and only if $\exists$ non-negative integers p and q such that $f^p(x) = f^q(y)$. It is quite easy to check that $\sim$ is an equivalence relation on X. We let $S_0 \in X/\sim$ be the equivalence class containing the point ξ.

The big idea of the proof is to view the elements of the set X as vertices and we draw a directed edge from the point x to $f(x)$, label this edge as $\Lambda(x)$. We are actually going to treat $\Lambda(x)$ as the length of the edge from x to $f(x)$. For our simplicity we call the points $x, f(x), f^2(x), \ldots$ as ancestors of x. Note that our graph does not contain any loop (except a trivial loop at the point ξ), because indeed if there exists a loop at some point $x_0 \in X$ we would get that $\exists\ m \in \mathbb{N}$ such that $f^m(x_0) = x_0$ but as f is stable and ξ is the unique fixed point of any iterate of f we get $x_0 = \xi$. Then as $f(\xi) = \xi$ we get that it is a trivial loop. So if we disregard the trivial loop at the point ξ our graph is simply a tree. The fact that our graph does not contains any cycle ensures that whenever two points are connected there exists an unique shortest path connecting them. Also note that in our tree two points x and y are connected means that they have a common ancestor, which basically means that they are related, i.e $x \sim y$. Thus each equivalence class represents an unique connected component of our tree.

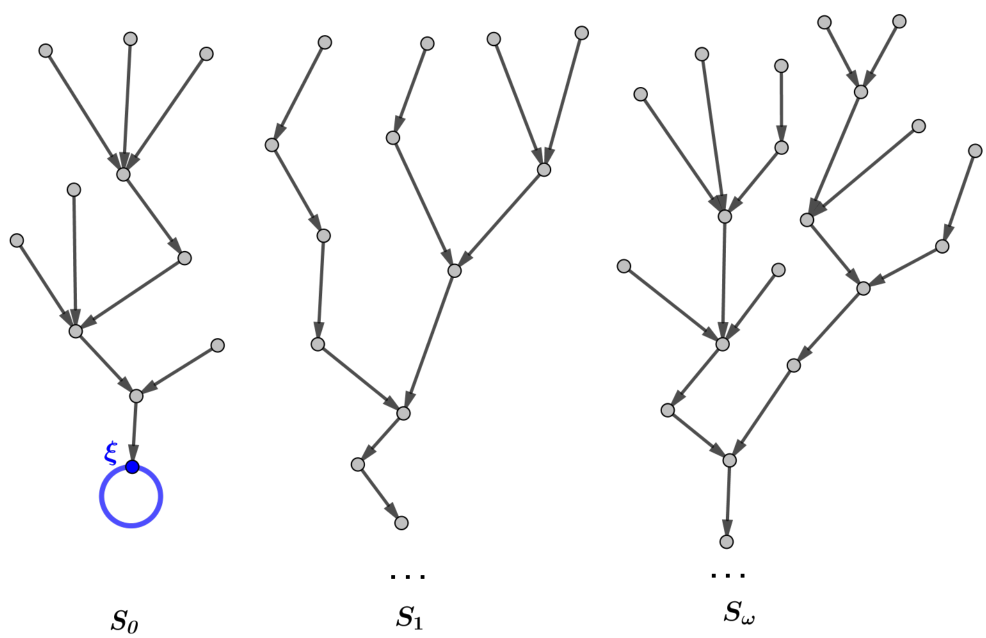


Figure 2: *It's almost a tree*

Now we need to define our metric d. Suppose $x \sim y$ and $x \neq y$ then there exists non-negative integers p and q (not both equal to 0) such that

$$f^p(x) = f^q(y) \tag{2}$$

i.e they have a common ancestor. We then trace the graph and find the nearest common ancestor of x and y, thus we are actually trying to find the least non-negative integers p and q that satisfy (2). Let

$p(x,y), q(x,y) :=$ denote minimum p and q respectively that satisfy (2)
and $\mathcal{N}(x,y) :=$ denote nearest common ancestor of x and y

And we let $d(x,y)$ be the length of the shortest path connecting x and y which is basically the sum of the length of the edges along the path $x \to \mathcal{N}(x,y) \to y$, which is simply the sum

$$d(x,y) = \sum_{j=0}^{p-1} \Lambda(f^j(x)) + \sum_{j=0}^{q-1} \Lambda(f^j(y)) \tag{3}$$

and if $x = y$ we simply define $d(x,y) = 0$.

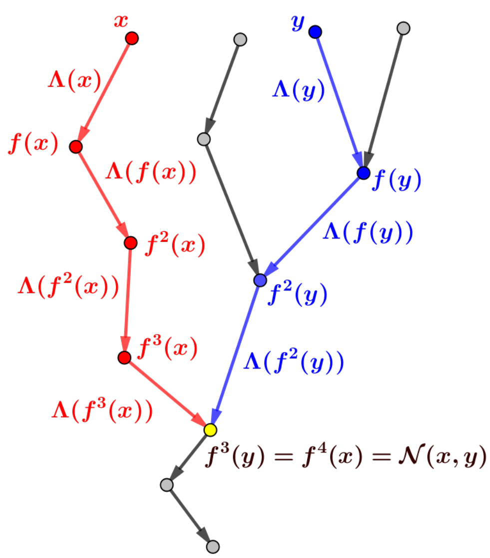


Figure 3: *defining* $d(x,y)$ *whenever* $x \sim y$

But we have not yet defined $d(x,y)$ when $x \nsim y$. For this case we define $f^{\infty}(x) = \xi \; \forall \; x \in X$, which intuitively means that every branch has a node at ξ if we go infinitely deeply. With this definition we can now define $d(x,y)$ even if $x \nsim y$, because now every pair of points x and y in the set X has a common ancestor ξ even if $x \nsim y$. Now when $x \nsim y$ we define

$$p(x,y) = \begin{cases} p(x,\xi) & \text{if } x \in S_0 \\ \infty & \text{if } x \notin S_0 \end{cases} \tag{4}$$

and similarly we have

$$q(x,y) = \begin{cases} q(\xi,y) & \text{if } y \in S_0 \\ \infty & \text{if } y \notin S_0 \end{cases} \tag{5}$$

Thus with all this definition when $x \nsim y$ we get $d(x,y)$ is given by

$$d(x,y) = \begin{cases} \sum_{j=0}^{p-1} \Lambda(f^j(x)) + \sum_{j=0}^{\infty} \Lambda(f^j(y)) & \text{if } x \in S_0 \text{ but } y \notin S_0 \\ \sum_{j=0}^{\infty} \Lambda(f^j(x)) + \sum_{j=0}^{q-1} \Lambda(f^j(y)) & \text{if } y \in S_0 \text{ but } x \notin S_0 \\ \sum_{j=0}^{\infty} \Lambda(f^j(x)) + \sum_{j=0}^{\infty} \Lambda(f^j(y)) & \text{if } x, y \notin S_0 \end{cases} \tag{6}$$

where $p := p(x, \xi)$ and $q := q(\xi, y)$.

But now to establish that d is a metric on X we first need to show that the infinite series $\sum_{j=0}^{\infty} \Lambda(f^j(x))$ converges for each $x \in X \setminus S_0$. The interesting result is whenever this infinite series converges for a suitable choice of Λ it can be shown that d is a metric on X and we regard the metric d as *Bessaga Brunner metric*. We need to define Λ wisely so that not just the sum converges but we also get f is a contraction map with respect to d.

Now this part is quite non-intuitive. We choose a representative element ζ_S for each equivalence class $S \in X/\sim$, *Axiom of Choice* guarantees such a set exists. Now we label the vertex ζ_S for each S by $\lambda(\zeta_S) = 0$ and then we add 1 to vertices as we move up in the tree and subtract 1 from the vertices as we move down in the tree, in this way we can determine λ uniquely on any vertex. And note that by the definition of our function $\lambda : X \to \mathbb{Z}$ we get

$$\lambda(f(x)) = \lambda(x) - 1 \ \forall \ x \in X \setminus \{\xi\} \tag{7}$$

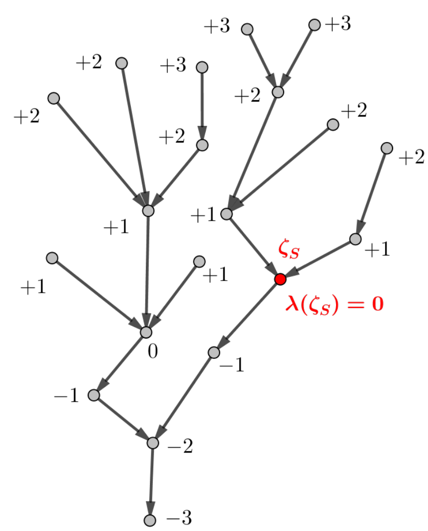


Figure 4: *Construction of* λ

And then by simple induction it can be shown that $\lambda(f^j(x)) = \lambda(x) - j, \ \forall \ j \in \mathbb{N}$. Now we define $\Lambda(x) = 2^{\lambda(x)} \ \forall \ x \in X$ (there is nothing special about the number 2 as we can take any positive real number strictly greater than 1). But then $d(x,y)$ reduces to

$$d(x,y) = \sum_{j=0}^{p-1} 2^{\lambda(x)-j} + \sum_{j=0}^{q-1} 2^{\lambda(y)-j} \tag{8}$$

where p, q may be infinite, but then even if p, q are infinite, the sum converges, hence $d(x, y)$ is now well defined $\forall\, x, y \in X$.

Lemma 3.0.1. *We have d is a metric on X further we have the metric space (X, d) is a complete metric space, and under this metric we have*

$$d(f(x), f(y)) \leq \frac{1}{2} d(x, y)\ \forall\, x, y \in X$$

i.e, f is a contraction map in the complete metric space (X, d).

Proof *(of d is a metric on X).*

Clearly $d : X^2 \to [0, \infty)$ is symmetric and $d(x, y) = 0$ if and only if $x = y$. So we only need to prove that d satisfies triangle inequality.

Let $x, y, z \in X$, we need to show that $d(x, y) \leq d(x, z) + d(z, y)$. We can view the path $x \to z \to y$ as a path from $x \to y$ via z. Now note that any path from $x \to y$ must pass through $\mathcal{N}(x, y)$ as it is their nearest common ancestor. But then as there is only an unique shortest path from $x \to \mathcal{N}(x, y)$ and $y \to \mathcal{N}(x, y)$ it means that the path $x \to z \to y$ contains the path $x \to \mathcal{N}(x, y) \to y$, and thus we get

$$d(x, z) + d(z, y) \geq d(x, \mathcal{N}(x, y)) + d(\mathcal{N}(x, y), y) = d(x, y)$$

Proof *(of f is a contraction map on (X, d)).*

Let $d(x, y) = \sum_{j=0}^{p-1} 2^{\lambda(x)-j} + \sum_{j=0}^{q-1} 2^{\lambda(y)-j}$ where p, q may be infinite then

$$\begin{aligned} d(f(x), f(y)) &= \sum_{j=0}^{p-2} 2^{\lambda(f(x))-j} + \sum_{j=0}^{q-2} 2^{\lambda(f(y))-j} \\ &= \sum_{j=0}^{p-2} 2^{\lambda(x)-j-1} + \sum_{j=0}^{q-2} 2^{\lambda(y)-j-1} \\ &\leq \frac{1}{2}\left(\sum_{j=0}^{p-1} 2^{\lambda(x)-j} + \sum_{j=0}^{q-1} 2^{\lambda(y)-j} \right) \\ &= \frac{1}{2} d(x, y) \end{aligned}$$

Hence f is a contraction map on (X, d). Now we need to show that the metric space (X, d) is complete.

Proof *(of (X, d) is a complete metric space).*

First we prove a simple claim.

Claim 3.0.2. $d(x, \xi) \leq 2^{\lambda(x)+1}$ *for each $x \in X$.*

Observe that $q(x, \xi) = 0$ for each $x \in X$ and using equation (4) we get that

$$d(x, \xi) = \sum_{j=0}^{p-1} 2^{\lambda(x)-j} \leq 2^{\lambda(x)} \left(\sum_{j=0}^{\infty} 2^{-j} \right) = 2^{\lambda(x)+1}$$

Lemma 3.0.3. *If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the metric space (M, ρ) and suppose we get that $\exists$ a subsequence $\{x_{n_k}\}$ converging to $x^* \in M$ then x_n also converge to x^* as $n \to \infty$.*

Proof of **Lemma 3.0.3** is quite trivial, so are not going into the proof. Now let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in (X, d), we have to show x_n converges in X. We divide the problem into two cases :

1. *If $\lambda\{x_n\}$ is unbounded below.*

   Then $\exists$ $\{x_{n_k}\}$ such that $\lambda(x_{n_k}) \to -\infty$. But then $d(x_{n_k}, \xi) \leq 2^{\lambda(x_{n_k})+1} \Rightarrow d(x_{n_k}, \xi) \to 0$. Hence, x_{n_k} converges to ξ, and as $\{x_n\}$ is Cauchy, thus we get $x_n \to \xi$.

2. *$\lambda(x_n) \geq M$, where M is a finite number.*

   Now if $x_m \neq x_n$ then at least one of $p(x_m, x_n), q(x_m, x_n)$ is positive, and so $d(x_m, x_n) \geq \min\left(2^{\lambda(x_m)}, 2^{\lambda(x_n)}\right) \geq 2^M$, but $\{x_n\}$ is Cauchy thus for sufficiently large m, n we must have $d(x_m, x_n) < 2^M$, hence x_m and x_n are not distinct after a certain $N_0 \in \mathbb{N}$.

   $\therefore$ (x_n) is eventually a constant sequence and hence convergent. □

The immediate implication of Bessaga's converse is if we want to proof the existence and uniqueness of fixed points of f^n $\forall$ $n \in \mathbb{N}$, then Banach's fixed point theorem is the universal way to do it.

4 Can we improve BCP ?

We first look at why contraction maps very essential in numerical analysis. Assume that f is a contraction map on (X, d) and let $\alpha = f_{<lips>}$ and ξ be the unique fixed point of f. Let $\{x_n\}_{n=0}^{\infty}$ be the defined by $x_0 \in X$ and $x_n = f\,(x_{n-1})$ $\forall$ $n \in \mathbb{N}$. Then we known that if $m > n$ we have

$$d(x_m, x_n) \leq \frac{\alpha^n}{1-\alpha} d(f(x_0), x_0)$$

taking $m \to \infty$ we get that

$$d(\xi, x_n) \leq \frac{\alpha^n}{1-\alpha} d(f(x_0), x_0) \tag{9}$$

Assume that we want to find a fixed point up to an "error" $(\varepsilon > 0)$, i.e. we want to find a point x^* such that

$$d(\xi, x^*) < \varepsilon$$

Equation (9) is important because it gives us an explicit value of n for which we would have $x_n = x^*$. Let $C = d(f(x_0), x_0)$ then observe that taking n such that

$$n > \frac{\ln\varepsilon + \ln(1-\alpha) - \ln C}{\ln\alpha} \tag{10}$$

fulfills our job. Thus we can find a bound on the number of iterations required to reach the our desired approximated fixed point.

Suppose we have some hypothesis $\mathcal{X}$ (which is not the Banach Fixed Point Theorem) on a function $f : X \to X$ which implies Banach Contraction Principle, i.e, f is stable. But then as f is stable by Bessaga's theorem we know that there exists a metric d on the set X

which makes f a contraction map, and also ensures that (X, d) is a complete metric space. But then we can simply work on the metric d and disregard our initial hypothesis $\mathcal{X}$. Thus even though any generalization or extension of Banach's theorem is quite useful and essential in the theoretical point of view, but in the practical realm of application it is not very useful. This is mostly because contraction map are easier for computational purposes. In the practical realm of application it is difficult to find the fixed point directly, but iterative nature of the contraction algorithm helps us to compute fixed point to very high degree of approximation in feasible amount of time.

Another thing to note is even though Bessaga's theorem implies the existence of the so called *'good-metric'* but finding an efficient metric that does our job is the challenging part. It is appropriate to say that *Bessaga's proof of Banach Contraction Principle, does not provide as with the most suitable metric rather it motivates us to find the most suitable metric that does our job*. Another drawback of Bessaga's result is that the *Bessaga Brunner Metric* does not convey any information about number of iteration required to reach the desired fixed point with respect to any pre-defined metric on the set X. This is why Meyer's Converse and Daskalakis Tzamos Zampetakis Converse are more stronger results than Bessaga's Converse.

References

[1] Saleh Almezel, Qamrul Hasan Ansari, and Mohamed Amine Khamsi. *Topics in fixed point theory*, volume 5. Springer, 2014.

[2] Tom Apostol. *Mathematical Analysis*. Addison-Wesley, 1974.

[3] Czełsaw Bessaga. On the converse of banach "fixed-point principle". *Colloquium Mathematicum*, 7:41–43, 1959.

[4] Constantinos Daskalakis, Christos Tzamos, and Manolis Zampetakis. A converse to banach's fixed point theorem and its cls-completeness. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, pages 44–50, 2018.

[5] Márton Elekes. On a converse to banach's fixed point theorem. *Proceedings of the American Mathematical Society*, 137(9):3139–3146, 2009.

[6] Jacek Jachymski. A short proof of the converse to the contraction principle and some related results. *Topological Methods in Nonlinear Analysis*, 15(1):179–186, 2000.

[7] Philip R Meyers. A converse to banach's contraction theorem. *J. Res. Nat. Bur. Standards Sect. B*, 71:73–76, 1967.

[8] Vittorino Pata. *Fixed point theorems and applications*, volume 116. Springer, 2019.

[9] Subrahmanyam P.V. Remarks on some fixed-point theorems related to banach's contraction principle. *Journal of Mathematical and Physical Sciences*, 8, 01 1974.

[10] Walter Rudin. *Principles of mathematical analysis*, volume 3. McGraw-hill New York, 1976.

[11] Eric Schechter. *Handbook of Analysis and its Foundations*. Academic Press, 1996.